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Symmetry analysis and exact invariant solutions for a class of energy-transport models of semiconductors

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Abstract

The symmetry classification of a class of energy-transport models for semiconductors is performed. Reduced systems and examples of exact invariant solutions are shown.

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1. Introduction

Continuum models for the description of charge carrier transport in semiconductors have attracted the attention of applied mathematicians and engineers in recent years on account of their applications in the design of electron devices. The *energy-transport* models for semiconductors (hereafter ET models) are macroscopic models that also take into account the thermal effects related to the electron flow through the crystal at variance with the popular *drift-diffusion* [1–3] models that are based on the assumption of isothermal motion.

The evolution equations are given by the balance equations for density and energy of the charge carriers, coupled to the Poisson equation for the electric potential. The pioneering models were proposed in [4, 5] on the basis of heuristic argument. A more systematic approach was followed in [6] starting from a harmonic expansion of the distribution function in the semiclassical kinetic framework. In these models there is the presence of some arbitrary functions as the mobilities, whose expression is based on the fitting of experimental data or Monte Carlo simulations. A recent derivation free of any arbitrary element has been obtained in [7] from the hydrodynamical model based on the extended thermodynamics [8, 9].

A partial analysis of some mathematical questions as to the existence and uniqueness of weak solutions can be found in [10, 11] for the model discussed in [6]. Here for the same model we perform a symmetry classification [12–16] and look for exact invariant solutions (the extension of this analysis to the model in [7] will be presented elsewhere).

The Lie point symmetries approach gives a systematic way to construct solutions for partial differential equations (PDEs). Looking for symmetries via the Lie infinitesimal criterion leads to the so-called *determining system* which is a linear PDEs system in the unknown coordinates of the invariance operator. The determining system, in general, is an over-determined system whose solution gives not only the coordinates of the invariance operator but also suggestions on the functional forms of the constitutive functions appearing in the equations. This leads to the group classification of the considered family of systems of PDEs. For each case present in the classification, we construct the optimal system of Lie subalgebra [12, 13, 17]. The main advantage of the use of the optimal system is that it minimizes the effort of searching for all the possible group-invariant solutions, collecting them into equivalence classes. The knowledge of invariant transformations allows us to reduce the number of independent variables. In particular, for problems in one spatial dimension, one can rewrite the original system of PDEs as a set of ordinary differential equations (ODEs) with considerable simplifications in searching for exact solutions.

The plan of the paper is as follows. In section 2, a brief review of the energy models is presented. In section 3, we perform the symmetry classification by determining the functional form of the constitutive functions for mobilities, energy relaxation time and doping profile, so that the balance equations admit symmetries. In particular, we recover for the mobilities the same expressions as those of the energy-transport model of Chen *et el* [4] (for a similar analysis of a class of *drift-diffusion* equations see [18]).

In section 4 the reduction to ODEs is shown and some examples of invariant exact solutions are presented for suitable doping profiles in the last section. We remark that these solutions can also be used to test numerical schemes and codes.

2. The mathematical model

The ET models for charge carriers in semiconductors are given by the balance equations for density and energy density for electrons (and also holes in the bipolar case), coupled to the Poisson equation for the electric potential

$$\frac{\partial n}{\partial t} + \operatorname{div} \boldsymbol{J} = 0 \tag{1}$$

$$\frac{\partial (nW)}{\partial t} + \operatorname{div} \boldsymbol{S} - \boldsymbol{J} \cdot \nabla \phi = nC_W \tag{2}$$

$$\lambda^2 \Delta \phi = n - c(\boldsymbol{x}) \tag{3}$$

where n is the electron density, $\boldsymbol{J}$ the electron momentum density, W the electron energy, $\boldsymbol{S}$ the energy flux density, nC_W the energy production, λ^2 the dielectric constant, ϕ the electric potential and $c(\boldsymbol{x})$ the doping concentration that is a given function of the position $\boldsymbol{x}$. Δ is the Laplacian operator and div the divergence operator. All the quantities are to be intended in a scaled form. The scaled variables have been obtained from the original ones by the transformations

$$\boldsymbol{x} \to \frac{\boldsymbol{x}}{\bar{l}} \qquad t \to \frac{t}{\bar{t}} \qquad n \to \frac{n}{\bar{c}} \qquad W \to \frac{W}{K_B T_L} \qquad \phi \to \frac{\phi}{U_T}$$

$$\boldsymbol{J} \to \frac{\boldsymbol{J}\bar{l}}{q\mu_0 U_T \bar{c}} \qquad \boldsymbol{S} \to \frac{\boldsymbol{S}\bar{l}}{q^2 \mu_0 U_T^2} \qquad C_W \to \frac{C_W}{q^2 \mu_0 U_T^2}$$

$$c \to \frac{c}{\bar{c}} \qquad \epsilon \to \frac{q\bar{c}\bar{l}^2 \epsilon}{U_T} = \lambda^2$$

where $\bar{t}$, $\bar{l}$ and $\bar{c}$ are typical values of time, length and doping density respectively, q is the elementary charge, $U_T = K_B T_L/q$ is the thermal voltage, with T_L the lattice temperature and K_B the Boltzmann constant, ϵ is the dielectric permittivity and μ_0 is the low field mobility.

The previous evolution equations can be derived as moment equations from the Boltzmann transport equation for electrons in semiconductors. The various ET models differ for the closure relations supplied for $\boldsymbol{J}$, W, $\boldsymbol{S}$ and C_W.

A class of ET models has been derived from the spherical harmonic expansion in [6]. The general form of the constitutive equations in the mobilities version is

$$W = \tfrac{3}{2}T \qquad C_W = -\frac{\tfrac{3}{2}(T - T_L)}{\tau_W(T)} \tag{4}$$

$$\boldsymbol{J} = -\nabla\left(\mu^{(1)}Tn\right) + \mu^{(1)}n\nabla\phi \tag{5}$$

$$\boldsymbol{S} = -\nabla\left(\mu^{(2)}T^2 n\right) + \mu^{(2)}Tn\nabla\phi. \tag{6}$$

T is the electron temperature, scaled according to $T \to \frac{T}{T_L}$, $T_L = 1$ is the scaled crystal temperature (taken as constant) and τ_W is the scaled energy relaxation time depending on T. $\mu^{(i)}$ are the electron mobilities that depend on T as well.

As particular cases we have:

- the model of Chen *et al* [4]

$$C_W = -\frac{\tfrac{3}{2}(T - T_L)}{\tau_0} \tag{7}$$

$$\boldsymbol{J} = -\mu_0\left(\nabla n - \frac{n}{T}\nabla\phi\right) \tag{8}$$

$$\boldsymbol{S} = -\frac{3}{2}\mu_0[\nabla(nT) - n\nabla\phi] \tag{9}$$

with $\mu_0 = 1$ the scaled low field mobility and τ_0 a positive constant; and

- the model of Lyumkis *et al* [5]

$$C_W = -\frac{2}{\sqrt{\pi}}\frac{T - T_L}{\tau_0 T^{1/2}} \tag{10}$$

$$\boldsymbol{J} = -\frac{2\mu_0}{\sqrt{\pi}}\left[\nabla\left(nT^{1/2}\right) - \frac{n}{T^{1/2}}\nabla\phi\right] \tag{11}$$

$$\boldsymbol{S} = -\frac{4\mu_0}{\sqrt{\pi}}[\nabla\left(nT^{3/2}\right) - nT^{1/2}\nabla\phi]. \tag{12}$$

3. The symmetry classification in the one-dimensional case

In the one-dimensional case the general energy transport model is given by the following class $\mathcal{C}$ of PDEs:

$$n_t + J_x = 0 \tag{13}$$

$$\frac{3}{2}(nT)_t + S_x + JE + \frac{3}{2}n\frac{(T - T_L)}{\tau_W(T)} = 0 \tag{14}$$

$$\lambda^2 E_x + n - c(x) = 0. \tag{15}$$

E is the electric field, which is related to the potential ϕ in the usual way, $E = -\phi_x$.

J and S are the relevant components of the electron momentum and energy flux. They are related to n, T and E through the constitutive relations

$$J = -\left[\left(\mu^{(1)}(T)Tn\right)_x + \mu^{(1)}(T)nE\right] \tag{16}$$

$$S = -\left[\left(\mu^{(2)}(T)T^2n\right)_x + \mu^{(2)}(T)TnE\right]. \tag{17}$$

We discuss the symmetry classification of the systems belonging to the class $\mathcal{C}$ of PDEs by the infinitesimal Lie method. The latter allows us to find the infinitesimal generator of the symmetry transformations and, at the same time, gives the functional dependence of the constitutive functions $\mu^{(1)}(T)$, $\mu^{(2)}(T)$, $\tau_W(T)$ and $c(x)$ for which the system does admit symmetries.

We consider the one-parameter Lie group of infinitesimal transformations in (x, t, n, T, E)-space given by

$$\hat{t} = t + \varepsilon\xi^1(x, t, n, T, E) + \mathcal{O}(\varepsilon^2) \tag{18}$$

$$\hat{x} = x + \varepsilon\xi^2(x, t, n, T, E) + \mathcal{O}(\varepsilon^2) \tag{19}$$

$$\hat{n} = n + \varepsilon\eta^1(x, t, n, T, E) + \mathcal{O}(\varepsilon^2) \tag{20}$$

$$\hat{T} = T + \varepsilon\eta^2(x, t, n, T, E) + \mathcal{O}(\varepsilon^2) \tag{21}$$

$$\hat{E} = E + \varepsilon\eta^3(x, t, n, T, E) + \mathcal{O}(\varepsilon^2) \tag{22}$$

where ε is the group parameter and the associated Lie algebra $\mathcal{L}$ is the set of vector fields of the form

$$X = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial n} + \eta^2\frac{\partial}{\partial T} + \eta^3\frac{\partial}{\partial E}. \tag{23}$$

One then requires that the transformation (18)–(22) leaves invariant the set of solutions of the system (13)–(15). In other words, one requires that the transformed system has the same form as the original one.

This yields an over-determined linear system of partial differential equations for the infinitesimals ξ^1, ξ^2, η^1, η^2 and η^3, which is called the *determining system*.

Note that even if the determining system is linear in the infinitesimals, the presence of the constitutive functions $\mu^{(1)}$, $\mu^{(2)}$, τ_W and c makes the equations very complicated.

The second prolongation of X we need is

$$\tilde{X} = X + \zeta_1^1\frac{\partial}{\partial n_t} + \zeta_2^1\frac{\partial}{\partial n_x} + \zeta_1^2\frac{\partial}{\partial T_t} + \zeta_2^2\frac{\partial}{\partial T_x} + \zeta_2^3\frac{\partial}{\partial E_x} + \zeta_{22}^1\frac{\partial}{\partial n_{xx}} + \zeta_{22}^2\frac{\partial}{\partial T_{xx}}$$

where the coefficients ζ_i^j and ζ_{22}^i ($j = 1, 2;\ i = 1, 2, 3$), after setting

$$(x^1, x^2) \equiv (t, x) \qquad (y^1, y^2, y^3) \equiv (n, T, E)$$

$$y_j^i = \frac{\partial y^i}{\partial x^j} \qquad y_{jk}^i = \frac{\partial^2 y^i}{\partial x^j \partial x^k} \quad (k = 1, 2)$$

$$\mathrm{D}_j = \frac{\partial}{\partial x^j} + y_j^i\frac{\partial}{\partial y^i} + y_{jk}^i\frac{\partial}{\partial y_k^i}$$

are given by

$$\zeta_j^i = \mathrm{D}_j\eta^i - y_1^i\mathrm{D}_j\xi^1 - y_2^i\mathrm{D}_j\xi^2$$

$$\zeta_{22}^i = \mathrm{D}_2\zeta_2^i - y_{12}^i\mathrm{D}_2\xi^1 - y_{22}^i\mathrm{D}_2\xi^2.$$

Table 1. Lie group classification. $\mu^{(1)} = \mu_0^{(1)} T^m$, $\mu^{(2)} = \mu_0^{(2)} T^m$. c_0, τ_0, p and q are constitutive constants.

Case	Forms of $\tau_W(T)$ and $c(\boldsymbol{x})$	Extensions of $L_{\mathcal{P}}$
I	τ_W arbitrary, $c = c_0$	$X_2 = \frac{\partial}{\partial x}$
II	$\tau_W = \frac{\tau_0(T-T_L)}{T^{(m+2)}}$ $c = c_0\,\mathrm{e}^{px}$ $p \neq 0$	$X_2 = -p(1+m)t\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + pn\frac{\partial}{\partial n} + pT\frac{\partial}{\partial T} + pE\frac{\partial}{\partial E}$
III	$\tau_W = \frac{\tau_0(T-T_L)}{T^{m+\frac{2(p+1)}{p+2}}}$ $c = c_0(x+q)^p$ $p \neq -2$	$X_2 = -[2m + p(1+m)]t\frac{\partial}{\partial t} + (x+q)\frac{\partial}{\partial x} + pn\frac{\partial}{\partial n}$ $+ (2+p)T\frac{\partial}{\partial T} + (1+p)E\frac{\partial}{\partial E}$

The determining system of (13)–(15) arises from the following invariance conditions

$$\tilde{X}(n_t + J_x) = 0$$
$$\tilde{X}\left(\frac{3}{2}(nT)_t + S_x + JE + \frac{3}{2}\frac{n(T-T_L)}{\tau_W(T)}\right) = 0$$
$$\tilde{X}\left(\lambda^2 E_x - c(x) + n\right) = 0$$

under the constraints that the variables n, T and E have to satisfy the equations (13)–(15).

The invariance conditions lead to the following result

$$\xi^1 = -[2ma_1 + (1+m)b_1]t + b_0 \tag{24}$$
$$\xi^2 = a_1 x + a_0 \tag{25}$$
$$\eta^1 = b_1 n \tag{26}$$
$$\eta^2 = (2a_1 + b_1)T \tag{27}$$
$$\eta^3 = (a_1 + b_1)E \tag{28}$$
$$(2a_1 + b_1)\left(T\mu_T^{(1)} + m\mu^{(1)}\right) = 0 \tag{29}$$
$$(2a_1 + b_1)\left(T\mu_T^{(2)} + m\mu^{(2)}\right) = 0 \tag{30}$$
$$(a_1 x + a_0)c_x - b_1 c = 0 \tag{31}$$
$$(2a_1 + b_1)\tau_{WT} + \frac{1}{T}\left[(2a_1 + b_1)\frac{T_L}{(T_L - T)} + 2ma_1 + (1+m)b_1\right]\tau_W = 0 \tag{32}$$

where a_0, a_1, b_0, b_1 and m are constants.

For $\mu^{(1)}$, $\mu^{(2)}$, τ_W and c arbitrary, from (24)–(32), we have that the *principal Lie algebra* $\mathcal{L}_{\mathcal{P}}$ of the system (13)–(15) is one dimensional and it is spanned by the operator

$$X_1 = \frac{\partial}{\partial t}. \tag{33}$$

Otherwise, we obtain

$$\mu^{(1)} = \mu_0^{(1)} T^m \qquad \mu^{(2)} = \mu_0^{(2)} T^m \tag{34}$$

with $\mu_0^{(1)}$ and $\mu_0^{(2)}$ constants. In this case the Lie algebras extend to one dimension $\mathcal{L}_{\mathcal{P}}$. The complete *Lie group classification* for the system (13)–(15) is reported in table 1.

Remark 1. If we set $m = -1$, one recovers the expressions for J and S of the model of Chen *et al*. Similarly if we set $m = -\frac{1}{2}$, one recovers the expressions for J and S of the model of Lyumkis *et al*. However in both cases there is a difference in the form of C_W.

Table 2. Non-trivial generators of the optimal systems. a is a real parameter.

Case	Generators of the optimal systems
I	$X_0 = a\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$
$\mathrm{II_a}$	$m = -1$
	$X_0 = a\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + pn\frac{\partial}{\partial n} + pT\frac{\partial}{\partial T} + pE\frac{\partial}{\partial E}$
$\mathrm{II_b}$	$m \neq -1$
	$X_0 = -p(1+m)t\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + pn\frac{\partial}{\partial n} + pT\frac{\partial}{\partial T} + pE\frac{\partial}{\partial E}$
$\mathrm{III_a}$	$2m + p(1+m) = 0$
	$X_0 = a\frac{\partial}{\partial t} + (x+q)\frac{\partial}{\partial x} + pn\frac{\partial}{\partial n} + (2+p)T\frac{\partial}{\partial T} + (1+p)E\frac{\partial}{\partial E}$
$\mathrm{III_b}$	$2m + p(1+m) \neq 0$
	$X_0 = -[2m + p(1+m)]t\frac{\partial}{\partial t} + (x+q)\frac{\partial}{\partial x} + pn\frac{\partial}{\partial n} + (2+p)T\frac{\partial}{\partial T} + (1+p)E\frac{\partial}{\partial E}$

Remark 2. In general, when a system of differential equations admits a Lie group $\mathcal{G}_r$ and its Lie algebra $\mathcal{L}_r$ is of dimension $r > 1$, one desires to minimize the search for invariant solutions by finding the nonequivalent branches of solutions. In fact if two subalgebras are similar, i.e. they are connected by a transformation belonging to the symmetry group (with Lie algebra $\mathcal{L}_r$), then their corresponding invariant solutions are connected by the same transformation. Therefore, it is sufficient to put into one class all similar subalgebras of a given dimension, say s, and select a representative from each class. The set of the representatives of all these classes is called an *optimal system of order s* [12]. In order to find all invariant solutions with respect to s-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*.

For the class $\mathcal{C}$ of PDEs an optimal system of Lie subalgebras has been obtained in [17]. The results are summarized in table 2.

4. Reduction to ODE systems

One of the advantages of the symmetry analysis is the possibility of finding solutions of the original system of PDEs by solving a system of ODEs. These systems of ODEs, called *reduced systems*, are obtained by introducing suitable new variables, determined as invariant functions with respect to the infinitesimal generator of the symmetry transformation.

On the basis of the infinitesimal generators of the optimal systems reported in table 2, we have the following reduced system.

4.1. Case I

The invariance conditions lead to

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}x}{1} \tag{35}$$

and give the similarity variable

$$z = t - ax \tag{36}$$

and the similarity solutions

$$n = \omega(z) \tag{37}$$

$$T = \chi(z) \tag{38}$$

$$E = \psi(z) \tag{39}$$

where ω, χ and ψ are arbitrary functions of the similarity variable z and must be solutions of the reduced system

$$\omega' + a\mu_0^{(1)}\left[(\omega\chi^m\psi)' - a\left(\omega\chi^{(1+m)}\right)''\right] = 0 \tag{40}$$

$$a\mu_0^{(2)}\left[\left(\omega\chi^{(1+m)}\psi\right)' - a\left(\omega\chi^{(2+m)}\right)''\right] - \mu_0^{(1)}\left[\omega\chi^m\psi - a\left(\omega\chi^{(1+m)}\right)'\right]\psi + \frac{3}{2}(\omega\chi)' + \frac{3\omega}{2\tau_W(\chi)}(\chi - T_L) = 0 \tag{41}$$

$$\lambda^2 a\psi' - \omega + c_0 = 0. \tag{42}$$

Here and in the following cases prime means differentiation with respect to z.

4.2. *Case II_a ($p \neq 0$)*

From the invariance conditions one has

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}n}{pn} = \frac{\mathrm{d}T}{pT} = \frac{\mathrm{d}E}{pE} \tag{43}$$

and obtains the similarity variable

$$z = t - ax \tag{44}$$

and the similarity solutions

$$n = \omega(z)\,\mathrm{e}^{px} \tag{45}$$

$$T = \chi(z)\,\mathrm{e}^{px} \tag{46}$$

$$E = \psi(z)\,\mathrm{e}^{px} \tag{47}$$

where ω, χ and ψ are arbitrary functions of the similarity variable z and must be solutions of the reduced system:

$$a^2\omega'' - \left[\frac{1}{\mu_0^{(1)}} + a\left(2p + \frac{\psi}{\chi}\right)\right]\omega' - \left[a\left(\frac{\psi}{\chi}\right)' - p\left(p + \frac{\psi}{\chi}\right)\right]\omega = 0 \tag{48}$$

$$a^2\mu_0^{(2)}(\omega\chi)'' - \left(\frac{3}{2} + 4ap\mu_0^{(2)}\right)(\omega\chi)' + \left(4p^2\mu_0^{(2)} - \frac{3}{2\tau_0}\right)\omega\chi + \mu_0^{(2)}[2p\omega\psi - a(\omega\psi)'] + \mu_0^{(1)}\left[-a\omega' + \left(p + \frac{\psi}{\chi}\right)\omega\right]\psi = 0 \tag{49}$$

$$\lambda^2(a\psi' - p\psi) - \omega + c_0 = 0. \tag{50}$$

4.3. *Case II_b ($p(1+m) \neq 0$)*

The invariance conditions give

$$-\frac{\mathrm{d}t}{p(1+m)t} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}n}{pn} = \frac{\mathrm{d}T}{pT} = \frac{\mathrm{d}E}{pE} \tag{51}$$

wherefrom the similarity variable

$$z = \frac{\ln(t)}{p(1+m)} + x \tag{52}$$

and the similarity solutions

$$n = \omega(z)t^{-\frac{1}{1+m}} \tag{53}$$
$$T = \chi(z)t^{-\frac{1}{1+m}} \tag{54}$$
$$E = \psi(z)t^{-\frac{1}{1+m}} \tag{55}$$

with ω, χ and ψ arbitrary functions of the similarity variable z and solutions of the reduced system

$$\mu_0^{(1)}\left[\left(\omega\chi^{(1+m)}\right)'' + (\omega\chi^m\psi)'\right] - \frac{\omega'}{p(1+m)} + \frac{\omega}{1+m} = 0 \tag{56}$$

$$\mu_0^{(2)}\left[\left(\omega\chi^{(2+m)}\right)'' + \left(\omega\chi^{(1+m)}\psi\right)'\right] + \mu_0^{(1)}\left[\left(\omega\chi^{(1+m)}\right)' + \omega\chi^m\psi\right]\psi - \frac{3}{2p(1+m)}(\omega\chi)' + \frac{3}{1+m}\omega\chi - \frac{3}{2\tau_0}\omega\chi^{(2+m)} = 0 \tag{57}$$

$$\lambda^2\psi' + \omega - c_0\,\mathrm{e}^{pz} = 0. \tag{58}$$

4.4. *Case III$_a$* $\left(m \neq -1,\, p = -\frac{2m}{m+1}\right)$

By proceeding as above, we have

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}x}{x+q} = \frac{\mathrm{d}n}{pn} = \frac{\mathrm{d}T}{(2+p)T} = \frac{\mathrm{d}E}{(1+p)E} \tag{59}$$

which gives the similarity variable

$$z = t - a\ln(x+q) \tag{60}$$

and the similarity solutions

$$n = \omega(z)(x+q)^p \tag{61}$$
$$T = \chi(z)(x+q)^{2+p} \tag{62}$$
$$E = \psi(z)(x+q)^{1+p} \tag{63}$$

where ω, χ and ψ are arbitrary functions of the similarity variable z and must solve the reduced system

$$\omega' + a\mu_0^{(1)}\left[-a\left(\omega\chi^{(1+m)}\right)'' + (2+p)\left(\omega\chi^{(1+m)}\right)' + (\omega\chi^m\psi)'\right] - (1+p)\mu_0^{(1)}\left[-a\left(\omega\chi^{(1+m)}\right)' + (2+p)\omega\chi^{(1+m)} + \omega\chi^m\psi\right] = 0 \tag{64}$$

$$\frac{3}{2}(\omega\chi)' + a\mu_0^{(2)}\left[-a\left(\omega\chi^{(2+m)}\right)'' + 2(2+p)\left(\omega\chi^{(2+m)}\right)' + (\omega\chi^{1+m}\psi)'\right] - 2(1+p)\mu_0^{(2)}\left[-a\left(\omega\chi^{(2+m)}\right)' + 2(2+p)\omega\chi^{(2+m)} + \omega\chi^{1+m}\psi\right] - \mu_0^{(1)}\left[-a\left(\omega\chi^{(1+m)}\right)' + (2+p)\omega\chi^{(1+m)} + \omega\chi^m\psi\right]\psi + \frac{3}{2\tau_0}\omega\chi = 0 \tag{65}$$

$$\lambda^2[a\psi' - (1+p)\psi] - \omega + c_0 = 0. \tag{66}$$

4.5. Case III_b $(2m + p(1 + m) \neq 0,\ p \neq -2)$

In this last case the invariance conditions read

$$-\frac{\mathrm{d}t}{[2m + p(1+m)]t} = \frac{\mathrm{d}x}{x+q} = \frac{\mathrm{d}n}{pn} = \frac{\mathrm{d}T}{(2+p)T} = \frac{\mathrm{d}E}{(1+p)E}. \tag{67}$$

One has the similarity variable

$$z = (x+q)t^{\frac{1}{2m+p(1+m)}} \tag{68}$$

and the similarity solutions

$$n = \omega(z)t^{-\frac{p}{2m+p(1+m)}} \tag{69}$$

$$T = \chi(z)t^{-\frac{2+p}{2m+p(1+m)}} \tag{70}$$

$$E = \psi(z)t^{-\frac{1+p}{2m+p(1+m)}}. \tag{71}$$

ω, χ and ψ depend on the similarity variable z and must solve the reduced system

$$\mu_0^{(1)}\left[\left(\omega\chi^{(1+m)}\right)'' + (\omega\chi^m\psi)'\right] - \frac{1}{2m + p(1+m)}(z\omega' - p\omega) = 0 \tag{72}$$

$$\mu_0^{(2)}\left[\left(\omega\chi^{(2+m)}\right)'' + \left(\omega\chi^{(1+m)}\psi\right)'\right] + \mu_0^{(1)}\left[\left(\omega\chi^{(1+m)}\right)' + \omega\,\chi^m\psi\right]\psi - \frac{3}{2[2m + p(1+m)]}\left[z(\omega\chi)' - 2p(1+p)\omega\chi\right] - \frac{3}{2\tau_0}\omega\chi^{m+\frac{2(p+1)}{p+2}} = 0 \tag{73}$$

$$\lambda^2\psi' + \omega - c_0 z^p = 0. \tag{74}$$

5. Invariant exact solutions

By solving the reduced systems of the previous section, one gets solutions of the original system of PDEs.

From the study of the reduced systems, the following classes of exact solutions have been found.

5.1. Case II_a $(p \neq 0)$

The reduced system (48)–(50) is autonomous and, under the condition $p^2 = \frac{3}{4\mu_0^{(2)}\tau_0}$, admits the constant solution

$$\omega = \lambda^2 K_0 + c_0 \qquad \chi = \frac{K_0}{p^2} \qquad \psi = -\frac{K_0}{p} \tag{75}$$

with K_0 an arbitrary constant. From (75) the stationary solution for the original PDE system (13)–(15)

$$n(x) = \left(\lambda^2 K_0 + c_0\right)\mathrm{e}^{px} \qquad T(x) = \frac{K_0}{p^2}\mathrm{e}^{px} \qquad E(x) = -\frac{K_0}{p}\mathrm{e}^{px} \tag{76}$$

is deduced.

Non-stationary solutions have also been found.

5.1.1.

$$n(t,x) = c_0\,\mathrm{e}^{px} - \frac{2\mu_0^{(1)}\lambda^2 K_0 p}{3\left(2\mu_0^{(2)} - \mu_0^{(1)}\right)}\,\mathrm{e}^{\alpha p x - \beta p^2 t} \tag{77}$$

$$T(t,x) = -\frac{1}{p} K_0\,\mathrm{e}^{\alpha p x - \beta p^2 t} \tag{78}$$

$$E(t,x) = K_0\,\mathrm{e}^{\alpha p x - \beta p^2 t} \tag{79}$$

where

$$\alpha = \frac{2\mu_0^{(1)}}{3\left(2\mu_0^{(2)} - \mu_0^{(1)}\right)} \qquad \beta = \frac{2\left(\mu_0^{(1)}\right)^2\left(6\mu_0^{(2)} - 5\mu_0^{(1)}\right)}{9\left(2\mu_0^{(2)} - \mu_0^{(1)}\right)^2}$$

under the conditions

$$p^2 = \frac{27\left(2\mu_0^{(2)} - \mu_0^{(1)}\right)^2}{2\mu_0^{(1)}\left[12\left(\mu_0^{(2)}\right)^2 + 16\mu_0^{(1)}\mu_0^{(2)} - 15\left(\mu_0^{(1)}\right)^2\right]\tau_0} \tag{80}$$

$$a = -\frac{3\left(2\mu_0^{(2)} - \mu_0^{(1)}\right)}{2\left(\mu_0^{(1)}\right)^2 p}. \tag{81}$$

5.1.2.

$$n(t,x) = c_0\,\mathrm{e}^{px} \qquad T(t,x) = -\frac{K_0}{p}\,\mathrm{e}^{-\frac{t}{\tau_0}} \qquad E(t,x) = K_0\,\mathrm{e}^{-\frac{t}{\tau_0}}. \tag{82}$$

5.2. Case II_b $(p(1+m) \neq 0)$

The reduced system has the solution

$$\omega(z) = k_1\,\mathrm{e}^{pz} \qquad \chi(z) = k_2\,\mathrm{e}^{pz} \qquad \psi(z) = k_3\,\mathrm{e}^{pz}$$

under the conditions

$$k_1 + \lambda^2 p k_3 - c_0 = 0 \tag{83}$$

$$k_2 p(m+2) + k_3 = 0 \tag{84}$$

$$\mu_0^{(2)}(m+3) + \mu_0^{(1)}(m+2)^2\left[(m+2)p - 1\right] - \frac{3}{2\tau_0 p^2} = 0. \tag{85}$$

This leads to the stationary solution of system (13)–(15)

$$n(x) = k_1\,\mathrm{e}^{px} \qquad T(x) = k_2\,\mathrm{e}^{px} \qquad E(x) = k_3\,\mathrm{e}^{px}. \tag{86}$$

5.3. Case III_a $\left(m \neq -1,\ p = -\frac{2m}{m+1}\right)$

The reduced system is again autonomous and admits the constant solution

$$\chi_0 = \left[\frac{3}{4(p+1)(p+2)\tau_0\mu_0^{(2)}}\right]^{\frac{1}{m+1}} \qquad \psi_0 = -(p+2)\chi_0$$

$$\omega_0 = c_0 + \lambda^2(p+1)(p+2)\chi_0$$

that gives the homogeneous solution to system (13)–(15)

$$n = \omega_0(x+q)^p \qquad T = \chi_0(x+q)^{p+2} \qquad E = \psi_0(x+q)^{p+1}. \tag{87}$$

5.4. Case III_b ($2m + p(1+m) \neq 0$, $p \neq -2$)

If we set $m = -1$ and $p = 1$, a class of stationary solutions is given by

$$n(x) = \left(c_0 - 2k_3\lambda^2\right)(x+q) \tag{88}$$
$$T(x) = k_2(x+q)^3 \tag{89}$$
$$E(x) = k_3(x+q)^2 \tag{90}$$

where k_2 is a real parameter and k_3 can take on the two values corresponding to the upper or lower sign

$$k_3 = \frac{\pm\sqrt{\left(9\left(\mu_0^{(2)}\right)^2 - 42\mu_0^{(1)}\mu_0^{(2)} + \left(\mu_0^{(1)}\right)^2\right)h_0^2k_2^2 + 6h_0k_2^{4/3}\mu_0^{(1)}} - h_0k_2\left(3\mu_0^{(2)} + \mu_0^{(1)}\right)}{2h_0\mu_0^{(1)}}$$

provided that the reality condition is satisfied.

For $p = 1$ but $m \neq -1$ we find the class of stationary solutions

$$n(x) = \left[c_0 + 2k(3m+4)\lambda^2\right](x+q) \tag{91}$$
$$T(x) = k(x+q)^3 \tag{92}$$
$$E(x) = -k(3m+4)(x+q)^2 \tag{93}$$

where k is a real parameter satisfying the relation

$$k^{1/3} - 6h_0\mu_0^{(2)}k^2(m+2) = 0.$$

Remark 3. The solutions found in the cases II_a and III_b for $m = -1$ are valid for the constitutive equations of fluxes of the model of Chen *et al*. The solutions obtained in the cases II_b, III_a and III_b, when specialized to $m = -\frac{1}{2}$, are valid for the constitutive equations of fluxes of the model of Lyumkis *et al*.

These exact solutions can be used as benchmarks for testing numerical codes for energy-transport models.

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